

Tasks and activities to enhance technological Pedagogical Mathematical Content Knowledge of teachers^{1, 2}

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Abstract From a sociocultural perspective, we examine activities generated by genres of tasks to understand how the tasks shape teachers knowledge of technology and mathematical content for teaching. The tasks and activities come from a professional development project that engages the cyberlearning system, Virtual Math Teams with GeoGebra. Working in teams, teachers enhance their understanding of dynamic geometry and how to engage in productive mathematical discussion. We theorize and discuss principles of our task design. We *explore* a task and the collaborative work of a team of teachers to illustrate relationships between the task design, productive mathematical discourse, and the development of new mathematics knowledge for the teachers. Implications of this work suggest further investigations into interactions between characteristics of task design and learners mathematical activity.

Key words

Collaboration, dynamic geometry, mathematical discourse, task design, technology, teachers' professional development.

Resumen³

Desde una perspectiva sociocultural, examinamos las actividades generadas por varios tipos de tareas para entender cómo las tareas dan forma al conocimiento de los docentes sobre la tecnología y el contenido matemático para la enseñanza. Las tareas y actividades provienen de un proyecto de desarrollo profesional que se acopla al sistema de aprendizaje cibernético que se llama Equipos Virtuales de Matemáticas con el software GeoGebra. Trabajando en equipos, los docentes mejoran su comprensión de la geometría dinámica y de la forma de participar en la discusión matemática productiva. Teorizamos y discutimos los principios de nuestro diseño de tareas. Exploramos una tarea y el trabajo en colaboración de un equipo de docentes para ilustrar las relaciones entre el diseño de las tareas, el discurso matemático productivo y el desarrollo de conocimientos nuevos de matemáticas para los docentes. Implicaciones de este trabajo sugieren nuevas investigaciones

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³ El resumen y las palabras clave en español fueron agregados por los editores.

sobre las interacciones entre las características de diseño de las tareas y la actividad matemática de los estudiantes.

Palabras clave

Colaboración, geometría dinámica, discurso matemático, diseño de tareas, tecnología, desarrollo profesional de docentes.

Mathematical tasks shape significantly what learners learn and structure their classroom discourse (Hiebert & Wearne, 1993). Such discussions when productive involve essential mathematical actions and ideas such as representations, procedures, relations, patterns, invariants, conjectures, counterexamples, and justifications and proofs about objects and relations among them. Nowadays, these mathematical objects and relations can be conveniently and powerfully represented in digital environments such as computers, tablets, and smartphones. Most of these environments contain functionality for collaboration. However, in such collaborative, digital environments, the design of tasks that promote productive mathematical discussions still requires continued theorization and empirical examination (Margolinas, 2013).

For mathematics teachers to support their students' engagement in productive mathematical discussions, they need require opportunities to enhance their technological pedagogical content knowledge. Their pedagogical interventions will emerging from a complex interplay among their knowledge of content, pedagogy, and technology (Mishra & Koehler, 2006). Teaching effectively with technology requires teachers to integrate these three domains of knowledge and to understand how each can influence their instructional decisions (Koehler & Mishra, 2008; Mishra & Koehler, 2006). We are interested in instructional tasks that shape learners' interaction with technology and moves learners towards deep mathematical understanding through productive mathematical discussions.

To theorize and investigate features of tasks that promote mathematical discussions, we are guided by this question: What features of tasks support productive discourse in collaborative, digital environments? Knowing these features will inform the design of rich tasks that promote mathematical discussions so that engaged and attentive learners build mathematical ideas and convincing forms of argumentation and justification in digital and virtual environments.

In virtual collaborative environments, the resources available to teachers to orchestrate collaboration and discourse among learners are different from those in traditional presentational classroom environments. The salient difference is that in presentational classroom environments the teacher is physically present, whereas in a virtual learning environment the teacher is artificially present; that is, the teacher exists largely as an artifact of digital tools. Consequently, the design of the tasks that are to be objects of learners' activities in virtual environments need to be constructed in ways that support particular learning goals such as productive mathematical discourse.

We share Sierpinska's (2004) consideration that "the design, analysis, and empirical testing of mathematical tasks, whether for purposes of research or teaching, is one of the most important responsibilities of mathematics education" (p. 10). In this paper, we focus on the design of tasks that embody particular intentionalities of an educa-

tional designer who aims to promote and support productive discourse in collaborative, digital environments. Our work employs a specific virtual environment that supports synchronous collaborative discourse and provides tools for mathematics discussions and for creating graphical and semiotic objects for doing mathematics. The environment, Virtual Math Teams (VMT), has been the focus of years of development by a team led by Gerry Stahl, Drexel University, and Stephen Weimar, The Math Forum @ Drexel University, and the target of much research (see, for example, Stahl, 2008; Stahl, 2009). Recently, research has been conducted on an updated VMT with a multiuser version of a dynamic geometry environment, GeoGebra, (Grisi-Dicker, Powell, Silverman, & Fetter, 2012; Powell, Grisi-Dicker, & Alqahtani, 2013; Stahl, 2013). Our tasks are designed for this new environment—VMTwG. Though the environment and its functionalities are not the specific focus of this paper, we will later describe some of its important features to provide context for understanding our design of tasks. Our focus here is to describe how we address challenges involved in designing tasks to orchestrate productive mathematical discourse in an online synchronous and collaborative environment. We first describe the theoretical foundation that guides our design of tasks to promote potentially productive mathematical discourse among small groups of learners working in VMTwG. Afterward, we describe our task-design methodology and follow with an example of a task along with the mathematical insights a small team of teachers developed discursively as they engaged with the task. We conclude with implications and suggestions areas for further research.

1. Theoretical Perspective

The theoretical foundation of our perspective on task design rests on a dialogic notion of mathematics (Gattegno, 1987), a view of the content of mathematics (Hewitt, 1999), what we call epistemic tools (Ray, 2013), and a sociocultural theory both of task and activity (Christiansen & Walther, 1986) and of instrument-mediated activity (Rabardel & Beguin, 2005).

Our notion of productive mathematical discourse rests on a particular view of what constitutes mathematics. From a psychological perspective, Gattegno (1987) posits that doing mathematics is based on dialog and perception:

No one doubts that mathematics stands by itself, is the clearest of the dialogues of the mind with itself. Mathematics is created by mathematicians conversing first with themselves and with one another. Still, because these dialogues could blend with other dialogues which refer to perceptions of reality taken to exist outside Man... Based on the awareness that relations can be perceived as easily as objects, the dynamics linking different kinds of relationships were extracted by the minds of mathematicians and considered *per se*. (pp. 13-14)

Mathematics results when a mathematician or any interlocutor talks to herself and to others about specific perceived objects, relations among objects, and dynamics involved with those relations (or relations of relations). For dialogue about these relations and dynamics to become something that can be reflected upon, it is important that they not be ephemeral and have residence in a material (physical or semiotic) record or inscrip-

tion. On the one hand, through moment-to-moment discursive interactions, interlocutors can create inscriptions and, during communicative actions, achieve shared meanings of them. On the other hand, inscriptions can represent encoded meanings that—based on previous discursive interactions—can grasp as they decode the inscriptions. Thus, inscriptive meanings and the specific perceived content of experience are dialectically related and mutually constitutive through discourse.

Through discourse, interlocutors among themselves construct or from others become aware of mathematical content. As Hewitt (1999) posits, mathematical content intended for learners to engage can be parsed into two essential categories. The first category pertains to content that is arbitrary in the sense that it refers to semiotic conventions such as names, labels, and notations. These conventions are historical and cultural, examples of which are the Cartesian axes, coordinates, names of coordinates, and notational rules. These conventions could have been otherwise and hence are arbitrary. Moreover, they cannot be constructed or appropriated through attentive noticing or awareness but rather must be known through memorization and association.

The second essential category concerns mathematical content that is necessary. These are ideas or properties that can be derived by attending to and noticing relations among objects as well as dynamics linking relations. For instance, when two planar, congruent circles have exactly two points of intersection, then an isosceles triangle can always be formed by choosing as its vertices the circles' centers and one intersection point. This conclusion, once known can be considered a cultural tool, is derivable, could not be otherwise, and therefore necessary. Relations among objects, dynamics of relations, and properties that can be worked out are necessary mathematical content. These particular mathematical ideas are historical and cultural tools and can be appropriated through awareness.

Whether particular necessary mathematical content is appropriated depends on awarenesses already possessed and attentive noticing. Awareness and noticing are elements that need to be accounted for in the design of tasks. As Hewitt (1999) notes

If a student does have the required awareness for something, then I suggest the teachers role is not to inform the student but to introduce tasks which help students to use their awareness in coming to know what is necessary. (p. 4)

Within this pedagogic paradigm, if students do not have requisite awareness, then they are invited to engage tasks that enable them to construct the required awareness. Constructing the awareness involves thinking mathematically. The teacher informs them of those cultural tools that are arbitrary and do not entail mathematical thinking and invite them to use existing awareness to notice and reason about necessary relations and relations of relations so as to appropriate new mathematical ideas through their discursive interaction.

To increase the probability that the discourse of interlocutors is mathematically productive, it is useful that they employ individual and collaborative discursive means to make sense of mathematical situations. For this purpose, we invite interlocutors to employ particular epistemic tools. That is, to ask questions of themselves and of their interlocutors that query what they perceive, how it connects to what they already know, and what they want to know more about it. Specifically, these tools include three

questions that interlocutors explicitly or implicitly engage: (1) What do you notice? (2) What does it mean to you? (3) What do you wonder about? The first and third questions come directly from work of The Math Form @ Drexel University (see, Ray, 2013). The second question is one that we have added. The purpose of these questions is to foster generative discussions within small groups of interlocutors that are grounded in their attention on perceivable, not necessarily visible, contents of experience that can be described as objects, relations among objects, and dynamics linking different relations. Using the epistemic tools, interlocutors' responses become public, relevant, and accountable. The idea is for interlocutors' to practice consciously these epistemic tools and over time become incorporated into their mathematical habits of mind.

The epistemic tools, among other things, are useful for enabling reflection on perceived infrastructural reactions of a dynamic geometry environment to interlocutors' actions in the environment. As they drag (click, hold, and slide) a base point of an object in a constructed figure, the environment redraws and updates information on the screen, preserving constructed geometrical relations among the figure's objects. This reaction to learners' dragging establishes a dialectical co-active relationship as the learner and the environment react to each other (Hegedus & Moreno-Armella, 2010). As learners attend to the environment's reaction, they experience and, since it responds in ways that are valid in Euclidean geometry, may become aware of underlying mathematical relations among objects such as dependencies.

Another role of the epistemic tools is to scaffold interlocutors' activity directed to understand and solve a mathematical task. We view tasks and activity from a sociocultural perspective. Within this perspective, Christiansen and Walther (1986) distinguish between task and activity in that "the *task* (the assignment set by the teacher) becomes the object for the student's activity" (p. 260). A task is the challenge or set of instructions that a teacher sets. An activity is the set of actions learners perform directed toward accomplishing the task. The activity is what students do and what they build and act upon such as material, mental, or semiotic objects and relations among the objects. The task initiates activity and is the object of students' activity.

Given the new digital, collaborative environments in which teaching and learning can occur, we find it theoretically useful to extend Christiansen and Walther's (1986) distinction of task and activity beyond analog environments: The purpose of a mathematical task in collaborative digital environments is to initiate and foster productive mathematical, discursive activity. The discursive activity is what learners communicate and do, what they build and act upon such as material, mental, or semiotic objects and relations. The digital, mathematical task is the object of learners' collective and coordinated activity.

Learners' activity directed toward a task is mediated by instruments. Before an instrument achieves its instrumental status, it is an artifact or tool. According to Rabardel and Beguin (2005) "the instrument is a composite entity made up of a tool component and a scheme component" (p. 442). The scheme component concerns how learners use the tool. Therefore, an instrument is a two-fold entity, part artifactual and part psychological. The transformation of an artifact into an instrument occurs through a dialectical process. One part accounts for potential changes in the instrument and the other accounts for changes in learners, respectively, instrumentalization and instrumentation. In instrumentalization, learners' interactions with a tool change how it is

used, and consequently, learners enrich the artifact’s properties. In instrumentation, the structure and functionality of a tool influence how learners use it, shaping, therefore, learners’ cognition (Rabardel & Beguin, 2005). The processes of instrumentalization, instrumentation, and activity as well as the interaction of learners with themselves and the task reside within a particular, evolving context that is cultural, historical, institutional, political, social, and so on (see Figure 1).

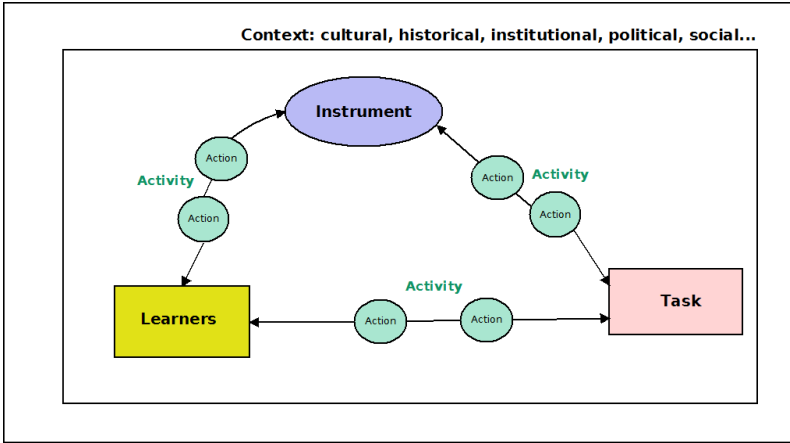


Figure 1: Relational model of learners engaged in instrument-mediated activity initiated by a task.

2. Task-design Methodology

Our methodology of task design embodies particular intentionalities for a virtual synchronous, collaborative environment, such as VMTwG, that has representation infrastructures (GeoGebra, a dynamic mathematics environment) and communication infrastructures (social network and chat features). The intentions are for mathematical tasks to be vehicles “to stimulate creativity, to encourage collaboration and to study learners’ untutored, emergent ideas” (Powell et al., 2009, p. 167) and to be sequenced so as to influence the co-emergence of learners instrumentation and building of mathematical ideas. To these ends, rooted in our theoretical perspective and sensitive to the infrastructural features of VMTwG, we developed and tested the following seven design principles for digital tasks that are intended to promote productive mathematical discourse by encouraging collaboration in virtual environments:

1. Provide a pre-constructed figure or instructions for constructing a figure.
2. Invite participants to interact with a figure by looking at and dragging objects (their base points) to notice how the objects behave, relations among objects, and relations among relations.

3. Invite participants to reflect on the mathematical meaning or consequence of what they notice.
4. Invite participants to wonder or raise questions about what they notice or the mathematical meaning or consequence of it.
5. Pose suggestions as hints or new challenges that prompt participants to notice particular objects, attributes, or relationships without explicitly stating what observation they are to make. Each hint has one or more of these three characteristics:
 - a) Suggest issues to discuss.
 - b) Suggest objects or behaviors to observe.
 - c) Suggest GeoGebra tools to use to explore relations, particularly dependencies.
6. Provide formal mathematical language that corresponds to awarenesses that they are likely to have explored and discussed or otherwise realized.
7. Respond with feedback based on participants' work in the spirit of the following:
 - a) Pose new situations as challenges that extend what participants have likely noticed, wondered, or constructed or that follow from an earlier task and that involve the same awarenesses or logical extensions of awarenesses they have already acquired.
 - b) Invite participants to revisit a challenge or a task on which they already worked to gain awareness of other relationships.
 - c) Invite participants to generalize noted relationships and to construct justifications and proofs of conjectures.
 - d) Invite participants to consider the attributes of a situation (theorem, figure, actions such as drag) in order to generate a "what if?" question and explore the new question.

The purpose of hints is to maintain learners' engagement with a task and to encourage them to extend what they know. The hints support participants' discourse by eliciting from them statements that reveal what they observe and what they understand about the mathematical meanings or consequences of their observations. The challenges are available to provide opportunities for learners to explore further by investigating new, related situations. Hidden initially, learners can reveal the hints and challenges by clicking a check box.

These design principles guided how we developed tasks in our research project, collaboration among investigators at Rutgers University and Drexel University. We employed VMTwG, which contains chat rooms for small teams to collaborate with tools for mathematical explorations, including a multi-user, dynamic version of GeoGebra. Team members construct geometrical objects and can explore them for relationships by dragging base points (see Figure 2). VMTwG records users' chat postings and

GeoGebra actions. The project participants are middle and high school teachers in New Jersey who have little to no experience with dynamic geometry environments and no experience collaborating in a virtual environment to discuss and resolve mathematics problems. The teachers took part in a semester-long professional development course. They met for 28 two-hour synchronous sessions in VMTwG and worked collaboratively on 55 tasks, Tasks 1 to 55.

Using our design principles, we developed dynamic-geometry tasks that encourage participants to discuss and collaboratively manipulate and construct dynamic-geometry objects, notice dependencies and other relations among the objects, make conjectures, and build justifications.

3. Task Example

We present the work of a team of two teachers on a task. The task, Task 10, is one that the research team posed. While the teachers worked on it, they posed a wondering that led us to provide feedback of type 7a, inviting them to explore that wondering. Our analysis reveals how using the epistemic tools the teachers noticed and discussed geometric relations and completed a construction task, wondered about the necessity of a foundational object of the construction, and in the following session resolved their wondering, all through the use of the epistemic tools.

In the fourth week of the professional development course, the team worked on Task 10. Employing procedures of Euclid’s second proposition (Euclid, 300 BCE/2002), the task engaged the team in constructing the copy of a line segment, without using the built-in compass tool, only using line segments, rays, and circles. The task also requested that they discuss dependencies and other relations among the objects (see Figure 2).

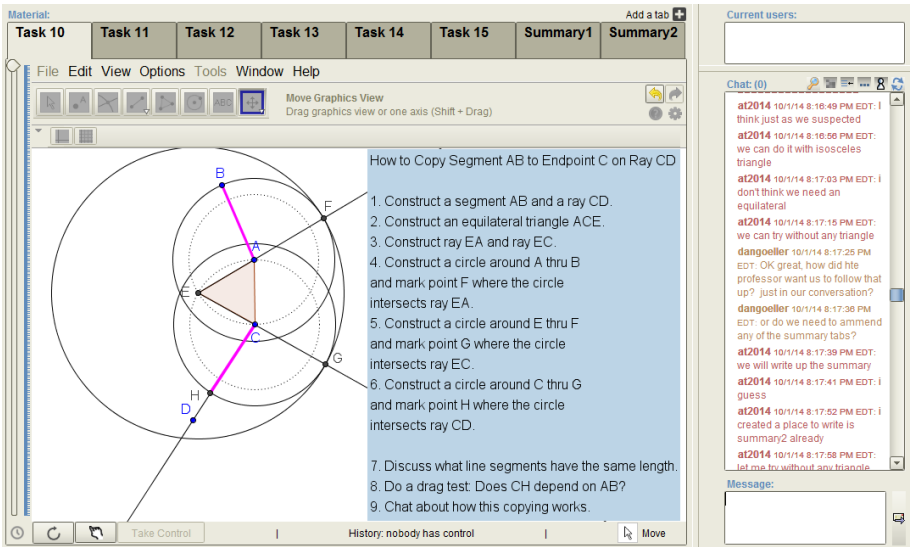


Figure 2: Task 10: Copying a line segment.

In the first synchronous session, the teachers successfully followed the construction instructions to copy segment AB onto ray CD. They used the epistemic tools to respond to this task and were attentive to co-active responses of VMTwG to their actions. In their noticings, they chatted about constructed dependencies and other relations among the geometric objects that they constructed. Below, an excerpt of the teachers' discussion illustrates their use of the epistemic tools and how they triggered productive mathematical discourse about a foundational aspect of the construction:

- 155 at2014: o what we wonder about
 156 at2014: lets talk about it before we move on
 157 at2014: i am still trying to understand so i am not quite sure whether
 the equilateral triangle is necessary
 158 at2014: o maybe it does
 159 dangoeller: i agree let's get the others done before sketching this one
 again
 160 at2014: to get that big circle
 161 at2014: ok
 162 dangoeller: that's a good question
 163 at2014: i am not sure why the equilateral triangle is necessary if it is
 at all
 164 dangoeller: it appears that it is, but the "why" behind it is unclear to me
 165 at2014: that would be the question for us to put in what we wondered
 about

In this excerpt, they employed the epistemic tools by wondering about whether an equilateral triangle is necessary in the construction procedure to copy a line segment (see lines 157, 163, and 164). In their session summary, they explicitly stated "We wonder whether the equilateral triangle is necessary or not and if it is necessary, why is it so." In our written feedback, their wondering encouraged us to invite them to explore it in their next synchronous session. In that session, they explored copying a length with an equilateral triangle, an isosceles triangle, and without using any specific type of triangle, which was essentially using a scalene triangle (see Figure 3).

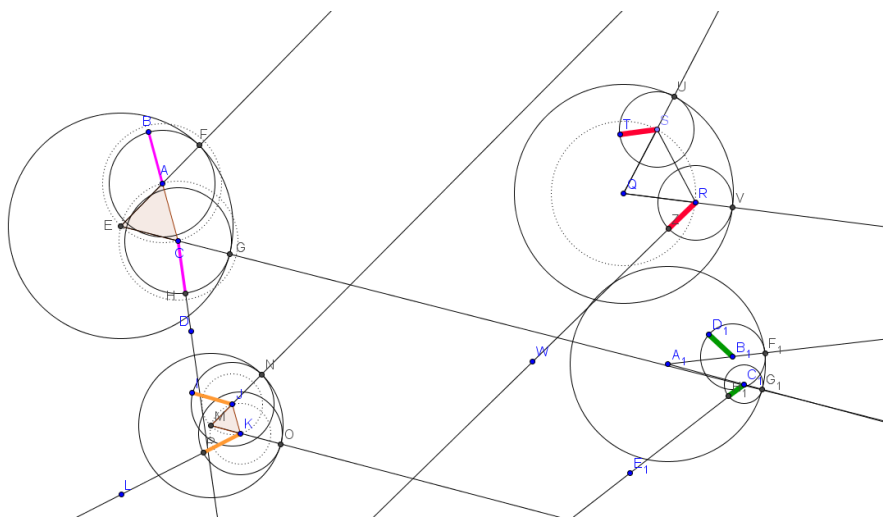


Figure 3: Teachers' investigation of minimal condition for copying a segment length.

The teachers wrote in their session summary that after conducting drag tests on their constructions, “we found out that if we want the length of one segment to be dependent on another, we need at least the isosceles triangle”. Their constructions in Figure 3 include copying a length with an equilateral triangle (lower left corner), using an isosceles triangle (top right corner), and “with no triangle” (lower right corner). They justified their findings by discussing the dependencies each construction has. They make the point that having an equilateral triangle “is only keeping points A and C apart a certain distance, and we can do without it.” That is, they demonstrated that to copy the length of the segment AB the distance between A and C is immaterial and that only two congruent sides of a triangle matter.

4. Discussion

Our focus was to describe how we address task design challenges to promote productive mathematical discourse among interlocutors working in an online synchronous environment. In the virtual environment, a teacher is present largely as an artifact of the environment’s digital tools and most specifically in the structure and content of tasks. An important feature of our task design is the questions of our epistemic tools since when collaborating interlocutors respond to them they generate propositional statements that can become the focus of their discussions. Their discussions are mathematically productive as their noticings, statements of meaning, and won-derings involve interpretations, procedures, patterns, invariants, conjectures, counter-examples, and justifications about objects, relations among objects, and dynamics linking relations.

Our guiding task-design principles aim to engage learners in productive mathematical activity through inviting them to explore figures, notice properties, reflect on relations, and wonder about related mathematical ideas. The design provides support through

hints and feedback to help learners with certain parts of the tasks. The tasks also include challenges that ask the participants to investigate certain ideas and extend their knowledge. The example provided above shows that the teachers moved from conjecture to justification through the use of our epistemic tools. They constructed ideas that were new to them. Further investigation is needed to understand how the task-design elements, the affordances of collaborative digital environments, and learners' mathematical discourse interact to shape the development of learners' mathematical activity and understanding.

References

- Christiansen, B., & Walther, G. (1986). Task and activity. In B. Christiansen, A. G. Howson & M. Otte (Eds.), *Perspectives in mathematics education: Papers submitted by members of the Bacomet group* (pp. 243–307). Dordrecht: Reidel.
- Euclid. (300 BCE/2002). *Euclid's elements* (T. L. Heath, Trans.). Santa Fe, NM: Green Lion.
- Gattegno, C. (1987). *The science of education: Part 1: Theoretical considerations*. New York: Educational Solutions.
- Grisi-Dicker, L., Powell, A. B., Silverman, J., & Fetter, A. (2012). Addressing transitional challenges to teaching with dynamic geometry in a collaborative online environment. In L. R. Van Zoest, J.-J. Lo & J. L. Kratky (Eds.), *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1024–1027). Kalamazoo, MI: Western Michigan University.
- Hegedus, S. J., & Moreno-Armella, L. (2010). Accommodating the instrumental genesis framework within dynamic technological environments. *For the Learning of Mathematics*, 30(1), 26–31.
- Hewitt, D. (1999). Arbitrary and necessary: Part 1 A way of viewing the mathematics curriculum. *For the Learning of Mathematics*, 19(3), 2–9.
- Hiebert, J., & Wearne, D. (1993). Instructional tasks, classroom discourse, and students learning in second-grade arithmetic. *American Educational Research Journal*, 30(2), 393–425.
- Koehler, M. J., & Mishra, P. (2008). What is technological pedagogical content knowledge? *Contemporary Issues in Technology and Teacher Education*, 9(1), 60–70.
- Margolinas, C. (2013). *Tasks design in mathematics education. Proceedings of ICMI Study 22*. Oxford, UK.
- Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A new framework for teacher knowledge. *Teachers College Record*, 108(6), 1017–1054.
- Powell, A. B., Borge, I. C., Floriti, G. I., Kondratieva, M., Koublanova, E., & Sukthankar, N. (2009). Challenging tasks and mathematics learning. In E. J. Barbeau & P. J. Taylor (Eds.), *Challenging mathematics in and beyond the classroom: The 16th ICMI study* (pp. 133–170). New York: Springer.
- Powell, A. B., Grisi-Dicker, L., & Alqahtani, M. (2013). Letramento matemático: Desenvolvendo as práticas colaborativas, matemáticas, e discursivas com tecnologia [Mathematical literacy: Development of collaborative, mathematical and discursive practices with technology] *XI Encontro Nacional de Educação Matemática, Educação Matemática: Retrospectivas e*

Perspectivas [XI National Conference of Mathematics Education, Mathematics Education: Retrospectives and Perspectives. Curitiba, Paraná.

- Rabardel, P., & Beguin, P. (2005). Instrument mediated activity: from subject development to anthropocentric design. *Theoretical Issues in Ergonomics Science*, 6(5), 429-461.
- Ray, M. (2013). Noticing and wondering *Powerful problem solving: Activities for sense making with the mathematical practices* (pp. 42-55): Heinemann.
- Sierpinska, A. (2004). Research in mathematics education through a keyhole: Task problematization. *For the Learning of Mathematics*, 24(2), 7-15.
- Stahl, G. (2008). Social practices of group cognition in virtual math teams. In S. Ludvigsen, A. Lund & R. Säljö (Eds.), *Learning in social practices: ICT and new artifacts—transformation of social and cultural practices*: Pergamon.
- Stahl, G. (2013). *Translating Euclid: Designing a human-centered mathematics*. San Rafael, CA: Morgan & Claypool.
- Stahl, G. (Ed.). (2009). *Studying virtual math teams*. New York: Springer.